

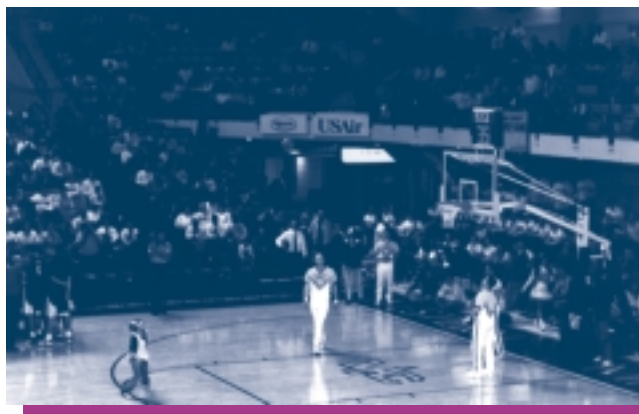
# Lesson 3

## Quadratic Models

Linear, exponential, and power models match the patterns of change in many important problems. By combining the algebraic expressions for those basic relations, you can build models for many other situations.

Among the most common examples of such combination models are equations in the form  $y = ax^2 + bx + c$ , the sum of power and linear rules. Those equations are called **quadratic models**. Specific numerical values of  $a$ ,  $b$ , and  $c$  give the rules that relate variables in specific situations.

Quadratic models help to describe the paths of many different kinds of flying objects. For example, at many basketball games there is a popular half-time contest to see who in the audience can make a long-distance shot.



### Think About This Situation

For a typical basketball shot, the ball's height (in feet) will be a function of time in flight (in seconds), modeled by an equation such as  $h = -16t^2 + 40t + 6$ .

- What sort of graph would you expect for the (*time in flight, height*) relation?
- How could you use the given rule relating height to time in flight to find when the shot might reach the height of the basket (10 feet)?
- How could you find the time when the ball would hit the floor, if it missed the basket entirely?

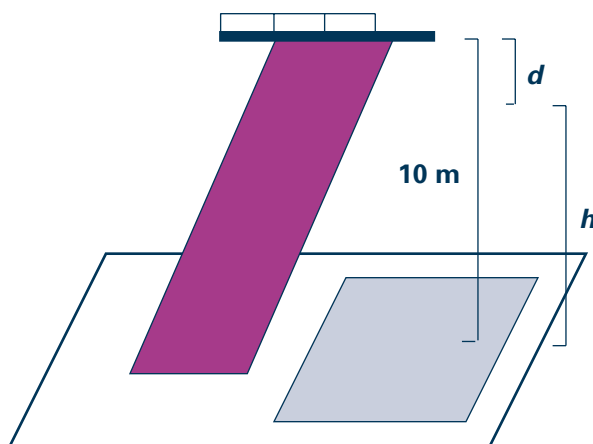
## INVESTIGATION 1 Going Up...Going Down

Spectator sports provide entertainment for many people, both young and old. One of the most beautiful, but scary, sports in the summer Olympic Games is platform diving. The divers jump from a tower that is 10 meters above the pool and perform twists and flips on their way down to the water. The time from take-off to landing is less than 2 seconds, and the divers are traveling very fast when they hit the water.



Gravity is the force pulling the divers down to the pool. The *distance fallen* is a function of *time in flight*. Assuming the diver's initial jump is not significantly high, distance fallen in meters can be estimated by the power rule  $d = 4.9t^2$ , where time is in seconds. (If you were to measure distance in feet instead, the power rule would be  $d = 16t^2$ . Can you figure out why?)

1. Another way to look at the diver's flight is to see how her height above the pool surface changes as time passes.



- a. Using the diagram on the previous page, explain how you would estimate the diver's height above the pool  $h$  at any time  $t$ . Keep in mind that the divers start at a height of 10 meters above the pool and travel a distance of  $4.9t^2$  meters in  $t$  seconds.
- b. Complete a table like the following one, which shows some sample time, distance, and height estimates.

Diving Estimates		
Time in Flight ( $t$ )	Distance Fallen ( $d$ )	Height Above Water ( $h$ )
0	0	10
0.25	0.306	$10.0 - 0.306 = 9.694$
0.50		
0.75		
1.00		
1.25		
1.50		
1.75		
2.00		

- c. About how long will it take a diver to reach the water?
2. Write an algebraic rule giving the diver's approximate height above the pool  $h$  as a function of time in flight  $t$ . Test your idea for the rule by using it in your calculator or computer software to produce a table of (*time in flight, height above water*) data. Compare that table with the results in Activity 1. Modify your rule if necessary.
  3. There are some high-diving competitions in which the tower is higher than 10 meters.
    - a. What rule would relate time and diver's height above the water if the tower were 20 meters high?
    - b. Estimate the time it would take a diver to hit the water from a 20-meter tower.
    - c. Compare the rules, tables, and graphs relating time and height for 10- and 20-meter platform dives. How are they similar and how are they different? What do the similarities and differences tell about the dives?

4. If you were going to make a high dive, you might wonder how fast you would be going when you hit the water. Suppose you jumped from a 20-m platform.
  - a. How far would you fall in the first 0.5 seconds? What would your average speed be in that time interval?
  - b. How far would you fall in the next 0.5 seconds (from 0.5 to 1.0 seconds)? What would your average speed be in that time interval?
  - c. How far would you fall in the next 0.5 seconds (from 1.0 to 1.5 seconds)? What would your average speed be in that time interval?
  - d. When would you hit the water and about how fast would you be falling at that time?
5. The gravity that pulls a platform diver down toward the water acts on all other falling objects near the surface of the Earth in the same way—that is, the distance fallen is  $d = 4.9t^2$ . Find algebraic rules to model the following situations and use those rules to estimate the time when each falling object hits the ground.
  - a. The relation between height  $h$  in meters and time in flight  $t$  in seconds of a marble dropped off a tall building from a height of 50 meters
  - b. The relation between height  $h$  in meters and time falling  $t$  in seconds of a baseball pop fly beginning at the maximum height of 25 meters

Gravity pulls falling objects toward the Earth's surface, but it also acts on things that appear to be flying upward. For example, a springboard diver bounces upward before being pulled back down to the water. In popular sports such as soccer and football, a ball is kicked into the air, only to have gravity pull it back down. In each case, the height at any time  $t$  is a function of two things: the initial upward velocity of the flying object and the force of gravity pulling in the opposite direction.



6. Suppose a diver bounces off a 3-meter springboard, moving upward at a speed of 4 meters per second. If there were no gravitational force pulling that diver back toward the pool surface, how would her height above the pool increase as time passed?
  - a. Complete this table of sample (*time in flight, height above water*) data.

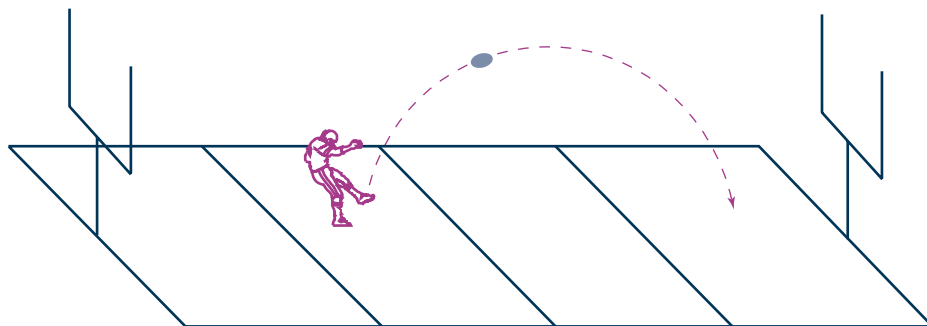
#### Springboard Diving (No Gravity)

<b>Time in Flight (sec)</b>	0	1	2	3	4	5	6
<b>Height Above Water (m)</b>	3						

- b. What algebraic rule would relate height above water  $h$  to time in flight  $t$ ?
- c. How would the table in Part a and the rule in Part b be different if the diver's initial speed was 2.5 meters per second?
- d. How would the table in Part a and the rule in Part b be different if the diver sprung off a 5-meter board instead of a 3-meter board?

7. Now think about how the real flight of the diver in Activity 6 results from a combination of three factors: initial height of the springboard, initial upward velocity produced by the spring of the board, and the force of gravity pulling the diver down toward the pool surface.
- Suppose the springboard is 3 meters high and the initial velocity of the diver is 4 meters per second upward. What algebraic rule would combine the three factors to give a relation between diver's height above the water  $h$  in meters and flight time  $t$  in seconds?
  - How would your answer to Part a change if the springboard were 5 meters high and the initial velocity were 2.5 meters per second upward?
8. For each situation given in Activity 7, use your calculator or computer software to make tables and graphs showing the expected relation between height above water and time in flight from takeoff to landing in the pool. Which do you think has the greater influence on the time available to perform twists and turns, the height of the springboard or the initial upward velocity of the diver? Explain your reasoning.

The physical forces and relationships that govern flight of a springboard diver apply to many other flying objects as well. Consider, for example, a punt by a football player.



9. The height of a football  $t$  seconds after a punt depends upon the initial height and velocity of the ball and on the downward pull of gravity. Suppose a punt leaves the kicker's foot at an initial height of 0.8 meters with initial upward velocity of 20 meters per second.
- Write an algebraic rule relating flight time  $t$  in seconds and height  $h$  in meters for this punt. Compare your rule with that of other groups and resolve any differences.
  - Compare your rule with the rule on page 265 relating height to time in flight of a basketball shot. In each case, identify the terms that represent force of gravity, initial velocity, and initial height of the ball.

10. Use the rule relating time in flight and height of the football to answer the following questions. In each case, explain how you arrived at your answer.
  - a. How does the height of the ball change as a function of time?
  - b. What is the maximum height the ball reaches, and when does that occur?
  - c. When does the ball return to the ground?
11. How will the rule change if the initial kick gives an upward velocity of only 15 meters per second? Use the new rule to answer the same questions posed in Activity 10.
12. Compare the rules and graphs that relate time and height of punts with initial upward velocities of 20 and 15 meters per second. How are they similar? How are they different?
13. How will the rule you formulated in Activity 9 change if the kick is a field-goal attempt, where the ball is held on the ground, rather than a punt, where the ball is dropped and kicked in the air?



## Checkpoint

The problems about platform divers and football punts involved models that were similar to, but a bit different from, the familiar power models. Compare the diving and punting models to the power model  $y = 4.9x^2$ .

- a. How is the graph of  $y = 10 - 4.9x^2$  similar to and different from the graph of  $y = 4.9x^2$  or  $y = -4.9x^2$ ?
- b. How is the graph of  $y = -4.9x^2 + 4x + 3$  similar to and different from the graph of  $y = 4.9x^2$  or  $y = -4.9x^2$ ?
- c. How are the patterns in tables for the models  $y = 10 - 4.9x^2$  and  $y = -4.9x^2 + 4x + 3$  similar to and different from those for the power models  $y = 4.9x^2$  or  $y = -4.9x^2$ ?
- d. How could you predict the patterns in tables and graphs by looking at the ways the symbolic rules for the new relations are built from the basic rule  $y = 4.9x^2$ ?

***Be prepared to share your observations and thinking with the class.***

## On Your Own

Refer to the rule your group developed in Activity 2.

- How would the rule change if the diving took place from a 10-meter platform and the gravity were the same as on the Moon, where distance fallen (in meters) is given by  $d = 0.83t^2$ ?
- Would you expect that the elapsed time for divers to hit the water from a 10-meter platform would be greater with Earth gravity or with Moon gravity? Explain your reasoning. Find what the elapsed time would be with Moon gravity.
- How does the rule for *Moon diving* from a 10-meter platform produce table and graph patterns of (*flight time, height above water*) data that are similar to and different from the rule for diving from a 10-meter platform with Earth gravity?
- How could the similarities and differences be predicted by studying the two rules?

## INVESTIGATION 2 Profit Prospects

Quadratic relations are also useful models in business situations. For example, if a concert promoter is planning a show by some popular band, research into costs and sales prospects could give a model predicting profit from the show as a function of chosen ticket prices. Suppose the promoter's research provides the data below. (Assume no other expenses or sources of income.)

Ticket Prices and Estimated Ticket Sales



Expenses

Band	\$6,000
Theater	\$1,500



1. The promoter used the (*ticket price, ticket sales*) data and a linear model to come up with the equation  $Ticket\ Sales = 4,000 - 250(Ticket\ Price)$ .
  - a. How could the promoter have come up with this equation?
  - b. Is the equation a good model of the relation between ticket price and probable ticket sales? Explain your reasoning.
  - c. How could the promoter use the equation or graph to predict the number of concert tickets sold?

2. From earlier work on modeling situations involving variables like those in this concert promotion problem, you should recall these basic relations:

$$Income = (Ticket\ Sales)(Ticket\ Price)$$

$$Profit = Income - Expenses$$

Using the letter names  $p$  for Ticket Price,  $S$  for Ticket Sales,  $I$  for Income, and  $P$  for profit, write equations giving the following.

- a. Ticket sales in terms of ticket price
  - b. Income in terms of ticket sales and ticket price
  - c. Profit in terms of income and expenses
3. The concert promoter decided that the key variable in planning was the ticket price  $p$  and wrote equations showing how ticket sales, income, and profit depended on ticket price. Using your answers to Activities 1 and 2, decide whether the equations (shown below) are correct.
    - a.  $S = 4,000 - 250p$
    - b.  $I = p(4,000 - 250p)$
    - c.  $I = 4,000p - 250p^2$
    - d.  $P = -250p^2 + 4,000p - 7,500$
  4. Use the relations among ticket price, ticket sales, income, and profit in Activity 3 to answer the following questions.
    - a. How many tickets probably will be sold if the ticket price is set at \$7?
    - b. What is the probable income from ticket sales if the ticket price is \$7?
    - c. What is the probable profit from the concert if ticket price is \$7?
  5. Use your graphing calculator or computer software to explore the relations among ticket price, ticket income, and concert profit to answer the following questions facing the promoter.
    - a. How does the estimate of income from ticket sales change as ticket prices from \$1 to \$15 are considered?
    - b. How does the estimate of profit from the concert change as ticket prices from \$1 to \$15 are considered?
    - c. For what ticket price or prices will the promoter break even on the concert?
    - d. What ticket price or prices will lead to maximum profit on the concert?
    - e. Look back at your answers to Parts a–d and explain why the results are or are not reasonable.



## Checkpoint

Summarize the key properties of the relationship between ticket price and profit for the concert and the ways that information is expressed in graphs and tables.

- What is the shape of the graph giving profit as a function of concert ticket price? What does that graph tell you about the relation between those variables?
- How would the pattern in the profit graph be displayed in a table of (*ticket price, profit*) values for ticket prices from \$1 to \$15?
- Where are break-even and maximum profit points on the graph? In the table?

***Be prepared to explain your responses to the class.***

## On Your Own

Suppose that surveys of interest for a different concert produced these data about the relation between ticket price and ticket sales.

### Concert Survey

<b>Ticket Price (\$)</b>	3	4	5	6	7
<b>Estimated Ticket Sales</b>	1,500	1,200	900	600	300

The promoter expects costs for this concert to total about \$3,000. Assume there are no other sources of income.

- Write algebraic equations that give the following.
  - Ticket sales  $S$  as a function of ticket price  $p$
  - Income  $I$  from ticket sales as a function of ticket price  $p$
  - Profit  $P$  as a function of ticket price  $p$
- Use your graphing calculator or computer software to explore the pattern of change in estimated ticket income and profit as different ticket prices from \$1 to \$10 are considered. Describe the patterns you find and explain why those patterns are or are not reasonable.
- Use the relations from Part a to find the following.
  - Income and profit expected from a ticket price of \$2
  - Ticket price or prices that will allow the concert to break even
  - Ticket price or prices that will lead to maximum income and to maximum profit. (These maxima may occur at different prices.)